Computer-readable proofs and dynamical systems

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Real and Complex Dynamical Systems
dedicated to Yulij Ilyashenko's 80th Birthday
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Different proof assistants are incompatible.

Example: Poincaré recurrence theorem

```
namespace MeasureTheory

variable {α : Type*} [MeasurableSpace α] {μ : Measure α} {f : α → α}

theorem exists_mem_image_mem [IsFiniteMeasure μ]
    (hf : MeasurePreserving f μ μ) (hs : MeasurableSet s) (hs' : μ s ≠ 0) :
    ∃ x ∈ s, ∃ m ≠ 0, f^[m] x ∈ s

structure Conservative (f : α → α) (μ : Measure α)
    extends QuasiMeasurePreserving f μ μ : Prop where
  exists_mem_image_mem : ∀ ⦃s⦄, MeasurableSet s → μ s ≠ 0 →
    ∃ x ∈ s, ∃ m ≠ 0, f^[m] x ∈ s

theorem MeasurePreserving.conservative [IsFiniteMeasure μ]
    (h : MeasurePreserving f μ μ) : Conservative f μ :=
  ⟨h.quasiMeasurePreserving, fun _ hsm h0 ↦ h.exists_mem_image_mem hsm h0⟩
```

Example: a proof

```
theorem exists_mem_iterate_mem_of_volume_lt_mul_volume
    (hf : MeasurePreserving f μ μ) (hs : MeasurableSet s)
    {n : ℕ} (hvol : μ (Set.univ : Set α) < n * μ s) :
    ∃ x ∈ s, ∃ m ∈ Set.Ioo 0 n, f^[m] x ∈ s := by
  have A : ∀ m, μ (f^[m] ⁻¹' s) = μ s := fun m ↦
    (hf.iterate m).measure_preimage hs
  have H : μ (univ : Set α) < ∑ m in Finset.range n, μ (f^[m] ⁻¹' s) := by
    simpa [A]
  obtain ⟨i, hi, j, hj, hij, x, hxi, hxj⟩ :
      ∃ i < n, ∃ j < n, i ≠ j ∧ (f^[i] ⁻¹' s ∩ f^[j] ⁻¹' s).Nonempty := by
    simpa using exists_nonempty_inter_of_measure_univ_lt_sum_measure
      μ (fun m _ ↦ (hf.iterate m).measurable hs) H
  wlog hlt : i < j generalizing i j
  · exact this j hj i hi hij.symm hxj hxi (hij.lt_or_lt.resolve_left hlt)
  refine ⟨f^[i] x, hxi, j - i,
    ⟨tsub_pos_of_lt hlt, lt_of_le_of_lt (j.sub_le i) hj⟩, ?_⟩
  rwa [← iterate_add_apply, tsub_add_cancel_of_le hlt.le]
```

Example: proof state

```
Goals (1)
α : Type u_1
inst : MeasurableSpace α
μ : Measure α
f : α → α
s : Set α
hf : MeasurePreserving f
hs : MeasurableSet s
n : ℕ
hvol : ↑↑μ univ < ↑n * ↑↑μ s
A : ∀ (m : ℕ), ↑↑μ (f^[m] ⁻¹' s) = ↑↑μ s
H : ↑↑μ univ < Finset.sum (Finset.range n) fun m ↦ ↑↑μ (f^[m] ⁻¹' s)
⊢ ∃ x ∈ s, ∃ m ∈ Ioo 0 n, f^[m] x ∈ s
```

Incomplete list of large formalization projects

in Coq, Isabelle, and Lean, and Mizar

Four color theorem Coq, Benjamin Werner and Georges Gonthier, 2005
Feit-Thompson Theorem Coq, Georges Gonthier, 2012
Complex analysis Isabelle/HOL, Thomas Hales
Proof of Kepler's Conjecture Isabelle and HOL Light, Thomas Hales and Co, 2014
Jordan Curve Theorem (HOL Light, Thomas Hales, 2007; Isabelle, Larry Paulson, 2017; Mizar, Artur Korniłowicz, 2007; Coq, Jean-François Dufourd, 2008)
Poincaré-Bendixson Theorem Isabelle/HOL, Fabian Immler, Yong Kiam Tan, 2020
Independence of the Continuum Hypothesis Lean, Floris van Doorn and Jesse Han, 2020
Connectedness of the Mandelbrot set Lean, Geoffrey Irving, 2023
Sphere eversion Lean, Floris Doorn, Patrick Massot, Oliver Nash, 2023
Galois Theory Lean, Thomas Browning, Patrick Lutz, 2022; Coq, Sophie Bernard, Cyril Cohen, Assia Mahboubi, Pierre-Yves Strub, 2021
Liquid Tensor Experiment (recent Peter Scholze's work) Lean, large team, 2022
Combinatorics: some recent proofs were formalized while the paper was still under review!

What's not formalized from the poll?

Before the talk, I distributed a "Guess what's formalized?" poll with 21 theorems. The following five from the list are not formalized yet (AFAIK).

- Cauchy-Kovalevskaya Theorem on existence of an analytic solution of an analytic PDE.
- Denjoy's theorem on rotation number.
- Herman-Yoccoz theorem on linearization of a circle diffeomorphism.
- Fermat's Last Theorem.
- Sard's Theorem.

Why bother? Correctness

Theorem (incorrect)

Let $f : V \to W$ be a map between complex normed spaces. Let U be a bounded nonempty set in V. If f is complex differentiable on U and is continuous on its closure, then the norm of $f(x)$ achieves its maximum on the closure of U at a point in the frontier of U.

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Theorem (correct)

Let $f : V \to W$ be a map between complex normed spaces. Suppose that V has finite positive dimension. Let U be a bounded nonempty set in V. If f is complex differentiable on U and is continuous on its closure, then the norm of $f(x)$ achieves its maximum on the closure of U at a point in the frontier of U.

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```
theorem Complex.exists_mem_frontier_isMaxOn_norm
{E : Type u} [NormedAddCommGroup E] [NormedSpace ℂ E]
[Nontrivial E] [FiniteDimensional ℂ E]
{F : Type v} [NormedAddCommGroup F] [NormedSpace ℂ F]
{f : E → F} {U : Set E} (hb : IsBounded U)
(hne : Set.Nonempty U) (hd : DiffContOnCl ℂ f U) :
∃ z ∈ frontier U, IsMaxOn (norm ∘ f) (closure U) z
```

Why bother? Correctness

As a consequence of my #Lean4 formalization project I have found a small (but non-trivial) bug in my paper!

– Terence Tao, Oct 23, 2023

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Improved search: If we have a huge library of formalized theorems, then one can index it and search it. Then you can see the exact assumptions that are guaranteed to work. We have `https://loogle.lean-fro.org/` and `https://theresanaiforthat.com/ai/moogle/`.

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Hide details Imagine moving boring details of a technical lemma to a computer-verifiable file and explain the idea behind it instead.

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Dependent type theory allows types like "pairs (a, b), where b has type $f(a)$".

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- This way the kernel is relatively small, so we have more reasons to trust it.
- Also, a buggy tactic can't make the system accept a faulty proof.

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- Almost 200000 lines of comments.
- A computer needs from 20 min to 3 hrs to verify it, depending on hardware.

My contributions (basic)

- the category of all categories;
- concrete categories;
- filter bases;
- convex hull;
- midpoint in an affine space;
- support of a function;
- complemented elements of a lattice;
- $f = \Theta(g)$;
- germ of a function at a filter;
- nontrivial filter;
- nonempty set;
- fixed and periodic points;
- involutive functions.

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- Tietze extension theorem.
- Continuous partitions of unity.
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- Split regular spaces from T_4 spaces.
- Define T_5 spaces and prove that a linear order is T_5.

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- Liouville numbers form a dense G_δ set of measure zero.

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About 1500 lines of code so far.

Happy Birthday Yulij Sergeevich!